Exam Sets & Numbers (part 2: integers and modular arithmetic),
February 2nd, 2024, 3:00pm–5:00pm,
Aletta Jacobshal 01 H22 - L8.

Write your name on every sheet of paper that you intend to hand in. Please provide **complete** *arguments for each of your answers. This part of the exam consists of* 2 *questions. You can score up to* 9 *points for each question, and you obtain* 2 *points for free. In this way you will score in total between* 2 *and* 20 *points.*

(1) A question regarding division and gcd's: let $a, b \in \mathbb{Z}_{>0}$, with $a \geq b$, and write $a = qb + r$ for integers $q \geq 1$ and $r \geq 0$.
  (a) [3 points] Show that $2^a - 1 = Q \cdot (2^b - 1) + 2^r - 1$ for some integer Q.
  (b) [2 points] Prove that $\gcd(2^a - 1, 2^b - 1) = \gcd(2^b - 1, 2^r - 1)$.
  (c) [2 points] Prove that $\gcd(2^a - 1, 2^b - 1) = 2^{\gcd(a,b)} - 1$.
  (d) [2 points] Prove that $(2^b - 1) \mid (2^a - 1) \iff b \mid a$.

(2) This problem considers, for integers $k \in \mathbb{Z}_{\geq 1}$, the numbers

$$b_k = \left(10^k + 5\right)/15$$

(and some modular arithmetic).
  (a) [1 point] Compute b_1 and b_2 and b_3 and b_4.
  (b) [2 points] Prove that $b_k \in \mathbb{Z}$, for every $k \in \mathbb{Z}_{\geq 1}$.
  (c) [1 point] Why is none of the integers b_k (for $k \geq 1$) divisible by 42?
  (d) [2 points] Show that $b_{k+1} = 10 \cdot b_k - 3$, for every $k \in \mathbb{Z}_{\geq 1}$.
  (e) [1 point] Explain why the sequence $(b_k \bmod n)_{k \geq 1}$ of elements in $\mathbb{Z}/n\mathbb{Z}$ is periodic, for any nonzero integer n.
  (f) [2 points] Today is $02 - 02 - 2024$. Show that $15 \cdot b_{2022024} \equiv 6 \bmod 11$ and that $b_{2022024} \equiv 7 \bmod 11$.

If you are only retaking the numbers part this side is all you need to complete, otherwise please turn over for part 1 on sets and do that part on a DIFFERENT piece of paper.